Derivation of functions τ and ρ from condition of unitarity

I. RELATIONSHIP BETWEEN FUNCTIONS τ AND ρ AND THE COUPLING η

The requirement that the sum of $\sim \hbar$-contributions vanish gives the condition

$$\frac{p_1}{m}\eta_{12}\frac{C_0}{\sqrt{\ddot{f}_0}} + \tau_{12}\frac{C_{\rm T}}{\sqrt{\ddot{f}_{\rm T}}}|v_1|^{-1} + \rho_{12}\frac{C_{\rm R}}{\sqrt{\ddot{f}_{\rm R}}}|v_1|^{-1} = 0 \tag{1}$$

where the additional factor $|v_1|^{-1} = \operatorname{sign} p_1 m/p_1$ is added after changing variable of integration $t_1 \to x_1$. The condition (1) is rewritten as

$$\eta_{12} = A_{12}\,\tau_{12} + B_{12}\,\rho_{12}, \tag{2}$$

where

$$A_{12} = -\frac{C_{\rm T}}{C_0}\sqrt{\frac{\ddot{f}_0}{\ddot{f}_{\rm T}}}\,\mathrm{sign}p_1, \tag{3}$$

$$B_{12} = -\frac{C_{\rm R}}{C_0}\sqrt{\frac{\ddot{f}_0}{\ddot{f}_{\rm R}}}\,\mathrm{sign}p_1. \tag{4}$$

II. A SIMILAR RELATIONSHIP, BUT FOR TRANSITION $2 \to 1$

A similar requirement that stems from transitions $2 \to 1$ gives the condition

$$\eta_{21} = A_{21}\,(-\tau_{21}) + B_{21}\,(-\rho_{21}), \tag{5}$$

where minus signs take into account opposite signs of corresponding integrals in the expansion.

III. CONSERVATION OF FLUX

Conservation of flux (without taking into account reflections) when moving $x \to x + \delta x$ means unitarity of the matrix that relates corresponding fluxes, $\mathsf{U} = \exp\left(\frac{i}{\hbar}\delta x \mathsf{T}\right)$, where

$$\mathsf{T} = \begin{pmatrix} |v_1| & 0 \\ 0 & |v_2| \end{pmatrix}^{1/2} \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix} \begin{pmatrix} |v_1| & 0 \\ 0 & |v_2| \end{pmatrix}^{-1/2}. \tag{6}$$

It means that the matrix T is self-adjoint, particularly

$$|v_1|^{1/2}\tau_{12}|v_2|^{-1/2} = |v_2|^{1/2}\tau_{21}^\star|v_1|^{-1/2}, \tag{7}$$

or $\tau_{21}^\star = \frac{p_1}{p_2}\tau_{12}$. If we take into account only reflections and consider exchange between waves moving in positive and negative directions, then similar condition gives relation $\rho_{21}^\star = \frac{p_1}{p_2}\rho_{12}$

IV. FINDING τ AND ρ

After substitutions of above relations into complex-conjugate of equation (5), we obtain

$$\eta_{12} = A_{21}^\star\frac{p_1}{p_2}\tau_{12} + B_{21}^\star\frac{p_1}{p_2}\rho_{12}, \tag{8}$$

where we used the fact that $\eta_{21}^\star = -\eta_{12}$. Solution of linear equations (2) and (8) is

$$\tau_{12} = \frac{B_{12}p_2 - B_{21}^\star p_1}{Dp_1}\eta_{12} \tag{9}$$

$$\rho_{12} = -\frac{A_{12}p_2 - A_{21}^\star p_1}{Dp_1}\eta_{12}, \tag{10}$$

where $D = A_{21}^\star B_{12} - A_{12}B_{21}^\star$.

In the limit of $\gamma \to \infty$, $A_{12} = A_{21} = B_{12} = -1$ and $B_{21} = 1$, because of negative sign of p_i in equation (4) when going back from negative to positive direction of propagation, and equations (9) and (10) give primitive-WKB functions

$$\tau_{12} = -\frac{p_1 + p_2}{2p_1}\eta_{12} \tag{11}$$

$$\rho_{12} = -\frac{p_1 - p_2}{2p_1}\eta_{12}. \tag{12}$$